A PRIME NUMBER THEOREM FOR THE MAJORITY FUNCTION

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Abstract. In the paper, the occurrence of zeros and ones in the binary expansion of the primes is studied. In particular the statement in the title is established. The proof is unconditional.

1. Introduction

Let $N = 2^n$ and identify $\{0, 1, \dots, N-1\}$ with $\{0, 1\}^n$ by binary expansion

$$x = \sum_{0 \le j < n} x_j 2^j \quad \text{with } x_j = 0, 1.$$

Assuming n odd, denote $f : \{0, 1\}^n \to \{0, 1\}$ the majority function. Motivated by a question of G. Kalai[Ka], we prove that f does not correlate with the primes, i.e.

Theorem 1. *Let Λ be the Von Mangoldt function. Then*

$$\sum_{1 \le x < N} \Lambda(x) f(x) \approx \frac{N}{2}. \tag{1.1}$$

Note that the majority function is a monotone Boolean function and it was proven in [B3] that the Moebius function does not correlate with any monotone Boolean function. The proof of his property uses the concentration of the Fourier-Walsh spectrum of monotone Boolean function on 'low levels'. More precisely, expanding

$$f(x) = \sum_{S \subset \{0, \dots, n-1\}} \hat{f}(S) w_S(x) \tag{1.2}$$

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with

$$w_S(x)=\prod_{j\in S}\varepsilon_j, \varepsilon_j=1-2x_j$$

the Walsh system on $\{0,1\}^n$, one exploits that

$$\sum_{|S|>n^{\frac{1}{2}+\varepsilon}}|\hat{f}(S)|^2$$

is small for monotone Boolean functions. This concentration is not sufficiently strong however to treat Λ instead of μ.

Recall that for the majority function, by symmetry, $\hat{f}(S)=\hat{f}(|S|)$ which obey

$$|\hat{f}(k)|^2\sim\binom{n}{k}^{-1}k^{-3/2}\quad\text{for}\quad k>0. \tag{1.3}$$

Hence

$$\sum_{|S|=k}|\hat{f}(S)|^2\sim k^{-3/2} \tag{1.4}$$

and

$$\sum_{|S|>k}|\hat{f}(S)|^2\lesssim k^{-1/2}. \tag{1.5}$$

Write

$$\sum_1^N\Lambda(x)f(x)=\frac{1}{2}\Big(\sum_1^N\Lambda(x)\Big)+N\sum_{0<|S|\le n}\hat{\Lambda}(S)\hat{f}(S). \tag{1.6}$$

Introducing some cutoff $n_0<n$, estimate the second term of (2.6) by

$$N\sum_{0<|S|\le n_0}|\hat{\Lambda}(S)|\;|\hat{f}(S)| \tag{1.7}$$

$+$

$$N\sum_{n_0<|S|\le n}|\hat{\Lambda}(S)|\;|\hat{f}(S)|. \tag{1.8}$$

Because primes are odd (except for the prime 2), for $S=(1,0,\dots,0)$,

$$\hat{\Lambda}(S)=\frac{1}{N}\sum_{x=1}^N\Lambda(x)(1-2x_1)=-\frac{1}{N}\sum_{x=1}^N\Lambda(x)\approx-1.$$

For $0 < |S| < \sqrt{n}$, $S \neq (1, 0, \dots, 0)$, it follows from [B2] that

$$|\hat{\Lambda}(S)| < e^{-c\sqrt{n}}. \tag{1.9}$$

Taking

$$n_0 \sim n^{\frac{1}{2} - \varepsilon} \tag{1.10}$$

the preceding permits to bound (1.7) by

$$O\Big(\frac{N}{\sqrt{n}} + N e^{-c\sqrt{n}} \sum_{k < n_0} \binom{n}{k}^{-\frac{1}{2}} k^{-\frac{3}{4}}\Big) = O\Big(\frac{N}{\sqrt{n}}\Big). \tag{1.11}$$

On the other hand, if we try to estimate (1.8) using L^2-norm, the tail estimate (1.5) implies

$$(1.8) \leq N\sqrt{n}\Big(\sum_{|S| > n_0} |\hat{f}(S)|^2\Big)^{\frac{1}{2}} \lesssim N\sqrt{n} n_0^{-1/4} \tag{1.12}$$

which is not conclusive, no matter how $n_0 \ll n$ is chosen.

Hence a more refined analysis is needed, involving more than just the low Fourier-Walsh spectrum of Λ.

In what follows, we will rely in particular on estimates related to those in the work of Mauduit-Rivat [M-R], where it was shown that Λ does not correlate with the parity function

$$\sigma(x) = e^{i\pi(\sum_{0 \leq j < n} x_j)} = w_{\{0,1,\dots,n-1\}}(x) \tag{1.13}$$

(rather than the majority function). See also [B1] from which we will borrow certain estimates.

Before going further, we point out the following easy consequence of [B2] on prescribing binary digits from the primes.

Theorem 2. *Let $\rho < \frac{4}{7}$. Then, with above notations, taking $r \sim n^\rho$, there are at least $O(2^{-r}\frac{N}{n})$ primes less than N satisfying*

$$\sum_1^n x_j > \frac{n}{2} + \frac{1}{3} r \tag{1.14}$$

and at least $O(2^{-r}\frac{N}{n})$ primes for which

$$\sum_1^n x_j < \frac{n}{2} - \frac{1}{3}r.$$

It follows indeed from [B2] that for $r < n^{\frac{4}{7}-}$, the set

$$\Omega_1 = \{p < N, x_0 = x_1 = \cdots = x_{r-1} = 1\}$$

satisfies

$$|\Omega_1| \sim \frac{N}{n}2^{-r}.$$

Since also for $1 \ll \Delta < \log n$

$$\Big|\Big\{x < N; x_0 = \cdots = x_{r-1} = 1 \text{ and } \Big|\sum_{r+1}^{n-1} x_j - \frac{n-r}{2}\Big| > \Delta\sqrt{n-r}\Big\}\Big| < e^{-c\Delta^2} N 2^{-r}$$

necessarily most elements of Ω_1 will satisfy

$$\Big|\sum_{j=r+1}^{n-1} x_j - \frac{n-r}{2}\Big| < O(\sqrt{n\log n})$$

and

$$\sum_{j=0}^{n-1} x_j > \frac{n+r}{2} - O(\sqrt{n\log n}).$$

The second part of the statement is proven similarly, considering the set

$$\Omega_0 = \{p < N; x_1 = \cdots = x_{r-1} = 0\}.$$

Note that it is essential for this argument that $r \gg n^{\frac{1}{2}}$.

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2. Symmetrization of the Von Mangoldt function

Returning to the proof of Theorem 1, we note that

$$\sum_1^N \Lambda(x) f(x) \equiv \langle \Lambda, f\rangle = \langle \Lambda_s . f\rangle$$

where Λ_s stands for the symmetrization of Λ under the permutation group of $\{0, 1, \ldots, n-1\}$. Thus

$$\Lambda_s = \sum_{k=1}^{n} \frac{\sum_{x \in \Omega_k} \Lambda(x)}{\binom{n}{k}} 1_{\Omega_k} \tag{2.1}$$

where $\Omega_k = \{x \in \{0,1\}^n; \sum x_j = k\}$.

The advantage of introducing Λ_s is a reduction of the L^2-norm. For $0 \leq \rho \leq 1$, denote T_ρ the usual convolution operator defined by

$$T_\rho w_S = \rho^{|S|} w_S$$

and which is a contraction on all L_p-spaces. Write

$$\langle \Lambda, f\rangle = \langle T_\rho \Lambda, f\rangle + \langle (1 - T_\rho)\Lambda_s, f\rangle = (2.2) + (2.3).$$

Then

$$(2.2) = \frac{1}{2} \sum_{1}^{N} \Lambda(x) + N \sum_{0<|S|\leq n} \rho^{|S|} \hat{\Lambda}(S) \hat{f}(S) \tag{2.4}$$

and estimate, recalling (1.9) the second term of (2.4) by

$$O\Big(\frac{N}{\sqrt{n}}\Big) + N \sum_{n_0<k\leq n} \rho^k k^{-3/4} \Big[\sum_{|S|=k} |\hat{\Lambda}(S)|^2\Big]^{\frac{1}{2}}$$

$$< O\Big(\frac{N}{\sqrt{n}}\Big) + N n^{\frac{1}{2}} \rho^{-n_0} < O\Big(\frac{N}{\sqrt{n}}\Big) \tag{2.5}$$

provided, cf (1.10), we set

$$\rho = 1 - n^{-\frac{1}{2}+2\varepsilon}. \tag{2.6}$$

To estimate (2.3), we decompose further

$$\Lambda_s = \Lambda_s' + \Lambda_s'' \tag{2.7}$$

denoting

$$\Lambda_s' = \sum_{|k-\frac{n}{2}|<\Delta\sqrt{n}} \frac{\sum_{k \in \Omega_k} \Lambda(x)}{\binom{n}{k}} 1_{\Omega_k}$$

with $\Delta \gg 1$ a parameter. Then

$$|\langle (1-T_\rho)\Lambda_s, f\rangle| \leq |\langle (1-T_\rho)\Lambda_s', f\rangle| + \|\Lambda_s''\|_1. \tag{2.8}$$

Estimate

$$|\langle (1-T_\rho)\Lambda_s', f\rangle| \leq \|\Lambda_s'\|_2 \ \|(1-T_\rho)f\|_2$$

where

$$\|\Lambda_s'\|_2 = \Bigg\{\sum_{|k-\frac{n}{2}|<\Delta\sqrt{n}} \frac{\big(\sum_{x\in\Omega_k}\Lambda(x)\big)^2}{\binom{n}{k}}\Bigg\}^{\frac{1}{2}} \leq$$

$$\sqrt{N}\Bigg[\max_{|k-\frac{n}{2}|<\Delta\sqrt{n}} \frac{\sum_{x\in\Omega_k}\Lambda(x)}{\binom{n}{k}}\Bigg]^{\frac{1}{2}} \lesssim$$

$$n^{\frac{1}{4}} e^{C\Delta^2}\Big[\max_k \sum_{x\in\Omega_k}\Lambda(x)\Big]^{\frac{1}{2}} \tag{2.9}$$

and, again form (1.3), (2.6)

$$\begin{aligned}\|(1-T_\rho)f\|_2 &\leq \sqrt{N}\Big[\sum_k (1-\rho^k)^2 k^{-3/2}\Big]^{\frac{1}{2}}\\ &\leq \sqrt{N}\Big[n_0^{-1/2} + \sum_{k\leq n_0} k^{1/2}(1-\rho)^2\Big]^{\frac{1}{2}} \lesssim n^{-\frac{1}{8}+2\varepsilon}\sqrt{N}.\end{aligned} \tag{2.10}$$

Hence

$$|\langle (1-T_\rho)\Lambda_s', f\rangle| \lesssim n^{\frac{1}{8}+2\varepsilon} e^{c\Delta^2}\Big\{\max_k \Big[\frac{1}{N}\sum_{x\in\Omega_k}\Lambda(x)\Big]\Big\}^{\frac{1}{2}} N. \tag{2.11}$$

Next

$$\|\Lambda_s''\|_1 = \sum_{|k-\frac{n}{2}|\geq\Delta\sqrt{n}} \Lambda(x).$$

Let $R \in \mathbb{Z}_+, R < \log n$ and estimate, again using the correlation estimates of Λ with low order Walsh functions

$$\sum_1^N \Lambda(x)\Big|\frac{n}{2} - \sum x_j\Big|^{2R} \leq \sum_1^N \Lambda(x)\Big|\sum_0^{n-1} \varepsilon_j\Big|^{2R} \\ \lesssim (CR)^R n^R N + (CR)^R\Big(\sum_{o<|S|\leq 2R} |\hat{\Lambda}(S)|\Big) \lesssim (CR)^R n^R N. \tag{2.12}$$

Therefore

$$\sum_{|\frac{n}{2}-\sum x_j|>\Delta\sqrt{n}} \Lambda(x) < e^{-c\Delta^2} N. \tag{2.13}$$

It remains to establish a bound on

$$\sum_{x\in\Omega_k} \Lambda(x) \tag{2.14}$$

for $|k - \frac{n}{2}| \leq \Delta\sqrt{n}$ in (2.11).

3. Distribution of the sum of the digits of the primes

Our remaining task is to bound (2.14) in the range $k = \frac{n}{2} + O(\sqrt{n})$. Take a bumpfunction η on $\mathbb{R}$ s.t. $\hat{\eta} \geq 0, \hat{\eta}(0) = 1$ and $\operatorname{supp}\eta \subset [-\frac{1}{2}, \frac{1}{2}]$ say.

Clearly

$$\sum_{x\in\Omega_k} \Lambda(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \Big[\sum_1^N \Lambda(x) e^{i\lambda(\sum_1^n x_j - k)}\Big]\eta(\lambda)d\lambda \tag{3.1}$$

and we evaluate

$$\sum_1^N \Lambda(x) U_\lambda(x) \tag{3.2}$$

where

$$U_\lambda(x) = e^{i\lambda(\sum\limits_0^{n-1} x_j)}. \tag{3.3}$$

This issue is very similar to the case of the Morse function ($\lambda = \pi$) considered by Mauduit-Rivat in [M-R]. Thus we will use the Vinogradov type I-II sum approach from [M-R]. In what follows, we will in fact rely on the presentation in [B1] (where the Moëbius function rather

than Λ is considered, but there is no essential difference here between these cases.)

The Fourier coefficients of U_λ obey an estimate

$$\left|\hat{U}_\lambda(k)\right| \lesssim e^{-c\lambda^2 n}. \tag{3.4}$$

The argument is similar to Lemma 2 in [B1]. In case of the Morse sequence $w_{\{0,1,\dots,n-1\}} = U_\pi$, one has in particular $\|\hat{U}_\pi\|_\infty < e^{-cn}$ which is stronger than (3.4) for small λ. This is the most significant difference compared with [B1].

Recall some terminology. Let $n = m_1 + m_2, M_1 = 2^{m_1}, M_2 = 2^{m_2}, m_1 \leq m_2$.

Type-II sums are of the form

$$\sum_{\substack{x^1\sim M_1\\ x^2\sim M_2}} a_{x_1} b_{x_2} U_\lambda(x^1.x^2) \tag{3.5}$$

where a_{x_1}, b_{x_2} are (arbitrary) bounded sequences (in fact obtained) as multiplicative convolutions of Λ and μ) and we may assume $M_1 > N^{\frac{1}{3}}$. For the Type-I sums, we set $b_{x_2} = 1$. Of course, the analysis of Type-II sums applies equally well to the Type-I sum but for the latter, also other considerations will be involved when M_1 is small.

We start by recalling the Type-II bound (2.31) from [B1], which in view of (3.4) becomes

$$|(3.5)| \lesssim N(L^{-c_1} + L^2 M_1^{-c_2} + L^{C_3} M_1^{-c\lambda^2}) \tag{3.6}$$

where c_1, c_2, C_3 are some constants, L a parameter (note that [B1] treats the case of an arbitrary Walsh function w_S, while for our purpose only the case $S = \{0, 1, \dots, n-1\}$ is of relevance).

Optimizing (3.6) in L gives a bound of the form

$$N M_1^{-c'\lambda^2}. \tag{3.7}$$

Next, according to [B1], (3.2') and (3.4), the following estimate on Type-I sums is gotten

$$M_1^2 M_2 \|\hat{U}_\lambda\|_\infty \lesssim N M_1\, e^{-c\lambda^2 n}. \tag{3.8}$$

Assuming $M_1 > N^{\frac{1}{3}}$, (3.7) gives a bound $Ne^{-c\lambda^2 n}$ on Type-II sums. The Type-I sums may be estimated using either (3.7) or (3.8), hence satisfy a bound $N.e^{-c\lambda^4 n}$, which is conclusive provided

$$\lambda > n^{-\frac{1}{4}+\varepsilon}. \tag{3.9}$$

The range (3.9) is not quite sufficient for our needs. Consequently assume

$$n^{-\frac{1}{2}+\varepsilon} < \lambda < n^{-\frac{1}{4}+\varepsilon} \tag{3.10}$$

and in view of the already available estimates (3.7), (3.8), also

$$c\lambda^2 n \lesssim m_1 < n^\varepsilon \lambda^{-2}. \tag{3.11}$$

Take

$$m_1 \ll m \ll n \tag{3.12}$$

to specify and decompose

$$x = (y, z) \in \{0,1\}^m \times \{0,1\}^{n-m}.$$

Write

$$U_\lambda(x) = e^{i\lambda(\sum_0^{m-1} y_j)}\ e^{i\lambda(\sum_m^{n-1} z_j)} = U(y)V(z).$$

Hence

$$\sum_{x^1\sim M_1} \Big| \sum_{x^2\sim M_2} U_\lambda(x^1.x^2)\Big| \leq \sum_z \sum_{x^1\sim M_1} \Big| \sum_{y\equiv -z(\operatorname{mod} x^1)} U(y)\Big|. \tag{3.13}$$

Some further manipulation of $U(y)$ is needed. Write $\varepsilon_j = 1 - 2y_j$ and

$$U(y) = e^{i\frac{\lambda}{2}m}\Big(\cos\frac{\lambda}{2}\Big)^m \prod_{j=1}^m (1 + i\varepsilon_j tg\,\frac{\lambda}{2}). \tag{3.14}$$

Expanding the last factor of (3.14) in the Walsh system

$$\prod_{j=1}^m \Big(1 + i\varepsilon_j tg\,\frac{\lambda}{2}\Big) = \sum_{k\leq k_1} (itg\,\frac{\lambda}{2})^k \sum_{|S|=k} w_S(\varepsilon) + \sum_{k_1<k\leq m} \cdots$$
$$= (3.15) + (3.16).$$

Taking

$$k_1 \sim \lambda^2 m \tag{3.17}$$

gives

$$\|(3.16)\|_2 < \sum_{m\geq k>k_1} |\lambda|^k \binom{m}{k}^{\frac{1}{2}} < 1$$

and the contribution of (3.16) in (3.13) is bounded by

$$\Big(\cos\frac{\lambda}{2}\Big)^m 2^m 2^{n-m} < e^{-c\lambda^2 m} N. \tag{3.18}$$

Next, in (3.15), expand

$$h = \sum_{|S|=k} w_S$$

in a regular Fourier series

$$h(y) = \sum_{r=0}^{2^m-1} \hat{h}(r) e\Big(\frac{ry}{2^m}\Big). \tag{3.19}$$

Fixing $0 \leq r < 2^m$, substituting in (3.13), we obtain

$$\begin{aligned} &\sum_{x^1\sim M_1} \Big| \sum_{y\equiv -z (\operatorname{mod} x^1)} e\Big(\frac{ry}{2^m}\Big)\Big| \lesssim \\ &\frac{2^m}{M_1} \sum_{x^1\sim M_1} 1_{[\|\frac{x^1 r}{2^m}\| < n\frac{M_1}{2^m}]}. \end{aligned} \tag{3.20}$$

Let $\delta > 0$ be another parameter and assume that

$$\sum_{x^1\sim M_1} 1_{[\|\frac{x^1 r}{2^m}\| < n\frac{M_1}{2^m}]} > \delta M_1. \tag{3.21}$$

By the pigeonhole principle, there is some $q' \lesssim \frac{1}{\delta}$ s.t. $\|\frac{q' r}{2^m}\| < n\frac{M_1}{2^m}$ and therefore we get

$$\frac{r}{2^m} = \frac{a}{q} + \theta \tag{3.22}$$

with

$$q \lesssim \frac{1}{\delta}, (a,q) = 1 \ \text{ and } \ |\theta| < \frac{nM_1}{q2^m}. \tag{3.23}$$

Assuming

$$\delta > 2^{-\frac{m}{2}} \tag{3.24}$$

it follows that $\|\frac{x^1 r}{2^m}\| \geq \|\frac{x^1 a}{q}\| - \frac{1}{2^{m-2m_1}} > \frac{\delta}{2}$, unless $x^1 a \equiv 0 (\operatorname{mod} q)$. If $x^1 a \equiv 0 (\operatorname{mod} q)$, $\|\frac{x^1 r}{2^m}\| = x^1 |\theta|$ and we obtain the condition

$$x^1 < \frac{n M_1}{2^m |\theta|}.$$

In view of (3.25), this implies that

$$|\theta| \lesssim \frac{n}{2^m \delta}. \tag{3.25}$$

Let $\frac{r}{2^m}$ satisfy (3.22) with

$$q < \frac{1}{\delta} \quad \text{and} \quad |\theta| \lesssim \frac{n}{2^m \delta}. \tag{3.25}$$

We estimate $\hat{w}_S(r)$. Thus, letting $\varphi = \frac{r}{2^m}$

$$\begin{aligned} \hat{w}_S(r) &= 2^{-m} \sum_{(x_0, \dots, x_{m-1}) \in \{0,1\}^m} e^{2\pi i \varphi (\sum_{j=0}^{m-1} 2^j x_j) + i\pi \sum_{j \in S} x_j} \\ &= 2^{-m} \prod_{j \notin S} (1 + e^{2\pi i 2^j \varphi}) \prod_{j \in S} (1 - e^{2\pi i 2^j \varphi}) \end{aligned}$$

and

$$\begin{aligned} |\hat{w}_S(r)| &= \prod_{j \notin S} |\cos \pi 2^j \varphi| \prod_{j \in S} |\sin \pi 2^j \varphi| \\ &\leq \prod_{\substack{j \notin S \\ j < m-J}} \left(\left| \cos 2\pi 2^j \frac{a}{q} \right| + \pi 2^j |\theta| \right) \prod_{\substack{j \in S \\ j < m-J}} \left(\left| \sin \pi 2^j \frac{a}{q} \right| + \pi 2^j |\theta| \right) \end{aligned} \tag{3.27}$$

with $1 \ll J \ll m$ to specify.

By (3.26), $2^j |\theta| \lesssim \frac{n}{\delta} 2^{-(m-j)} \lesssim \frac{n}{\delta}.2^{-J}$ and we take

$$J \sim \log \frac{1}{\delta} + k \tag{3.28}$$

as to ensure that

$$|\hat{w}_S(r)| \leq \prod_{\substack{j \notin S \\ j < m-J}} \left| \cos 2\pi 2^j \frac{a}{q} \right| \prod_{\substack{j \in S \\ j < m-J}} \left| \sin \pi 2^j \frac{a}{q} \right| + \delta. \tag{3.29}$$

Recall that $|S| = k \leq k_1 \sim \lambda^2 m$. It follows that there is an interval $\{j_0, \dots, j_1 - 1\}$ in $\{0, \dots, [\frac{m}{2}]\}$ of size

$$j_1 - j_0 > \frac{m}{2k_1} \tag{3.30}$$

which is disjoint from S. The first factor in (3.29) is then majorized by

$$\begin{aligned}\prod_{j\in I} \Big| \cos 2\pi 2^j \frac{a}{q} \Big| &= \frac{1}{2^{j_1-j_0}} \Big| \sum_{u=0}^{2^{j_1-j_0}-1} e^{2\pi i 2^{j_0}\frac{a}{q}} \Big| \\ &\leq \frac{q}{2^{j_1-j_0}} < \frac{1}{\delta 2^{\frac{m}{2k_1}}}\end{aligned} \tag{3.31}$$

provided q is not a power of 2. On the other hand, if q is a power of 2, then $\sin \pi 2^j \frac{a}{q} = 0$ for $j \gtrsim \log \frac{1}{\delta}$ and we conclude that

$$|\hat{w}_S(r)| < \delta + \frac{1}{\delta 2^{\frac{m}{2k_1}}} < \delta + \frac{1}{\delta} e^{-c\lambda^{-2}} \tag{3.32}$$

except if $S \subset \{0, 1, \dots, J\} \cup \{m - J, \dots, m - 1\}$.

Consequently, the contribution of the k-term of (3.15) in (3.13) may be estimated as follows

$$2^n \Big(\cos \frac{\lambda}{2} \Big)^m \Big| tg \frac{\lambda}{2} \Big|^k \Big\{ \|\hat{h}\|_1 \, \delta + \binom{m}{k} \Big(\delta + \frac{1}{\delta} \, e^{-c\lambda^{-2}} \Big) + \binom{2J}{k} \max_{|S|=k} \|\hat{w}_S\|_1 \Big\}. \tag{3.33}$$

with J given by (3.28).

Making a suitable approximation of the step-function by Fourier-truncation (cf. [B1] for details), with an L^1-error at most m^{-k} say, we ensure that

$$\|\hat{w}_S\|_1 < (ck \log n)^k \tag{3.34}$$

and hence

$$\|\hat{h}\|_1 < \binom{m}{k} (ck \log n)^k. \tag{3.35}$$

Substituting (3.34), (3.35) in (3.33), we find

$$(3.33) < 2^n e^{-c\lambda^2 m} \Big\{ m^{2k} \delta + m^k \delta^{-1} e^{-c\lambda^{-2}} + \Big(1 + \frac{c \log \frac{1}{\delta}}{k} \Big)^k (ck\lambda \log n)^k \Big\}. \tag{3.36}$$

Taking $\delta = m^{-2k}$ gives

$$\begin{aligned}(3.33) &< 2^n\, e^{-c\lambda^2 m}\Big(1 + m^{3k_1} e^{-c\lambda^{-2}} + (Ck_1(\log n)^2\lambda)^k\Big)\\ &< 2^n\, e^{-c\lambda^2 m}\Big(1 + e^{c(\log n)\lambda^2 m - c\lambda^{-2}} + (C(\log n)^2\lambda^3 m)^k\Big).\end{aligned} \tag{3.37}$$

Recalling (3.10)-(3.12), take

$$m = \frac{c}{(\log n)^2}\min(\lambda^{-3}, n). \tag{3.38}$$

Then

$$(3.37) < 2^n e^{-c\lambda^2 m} < 2^n e^{-c(\log n)^{-2}\min(\lambda^{-1}, \lambda^2 n)} \tag{3.39}$$

which gives a bound for the (3.15)-contribution to (3.13).

Thus we proved that if $n^{-\frac{1}{2}+\varepsilon} < \lambda < n^{-\frac{1}{4}+\varepsilon}$ and $\lambda^2 n \lesssim m_1 < \frac{n^\varepsilon}{\lambda^2}$, then

$$(3.13) < Ne^{-n^\varepsilon}. \tag{3.40}$$

Summarizing, we conclude that (3.2) may certainly be bounded by $\frac{N}{n}$ provided $\lambda > n^{-\frac{1}{2}+\varepsilon}$.

Consequently, substituting in (3.1) gives

$$\sum_{x\in\Omega_k}\Lambda(x) < Nn^{-\frac{1}{2}+\varepsilon}. \tag{3.41}$$

4. Conclusion of the proof of Theorem 1

Substitution of (3.41) in (2.9) gives

$$\|\Lambda_s'\|_2 < e^{C\Delta^2} n^\varepsilon N \tag{4.1}$$

and in (2.11)

$$|\langle(1-T_\rho)\Lambda_s', f\rangle| < e^{C\Delta^2} n^{-\frac{1}{8}+3\varepsilon} N. \tag{4.2}$$

Recalling (2.5) and (2.13), we proved that

$$\begin{aligned}\langle\Lambda, f\rangle &< O\Big(\frac{N}{\sqrt{n}}\Big) + e^{C\Delta^2} n^{-\frac{1}{8}+3\varepsilon} N + e^{-c\Delta^2} N\\ &< n^{-c} N\end{aligned} \tag{4.3}$$

for some constant $c > 0$, by suitable choice of Δ.

Hence, Theorem 1 holds in the more precise form

$$\sum_1^N \Lambda(x)f(x) = \frac{N}{2} + O(n^{-c}N). \tag{4.4}$$

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